JUNIOR MATHEMATICIAN

(A journal for students)

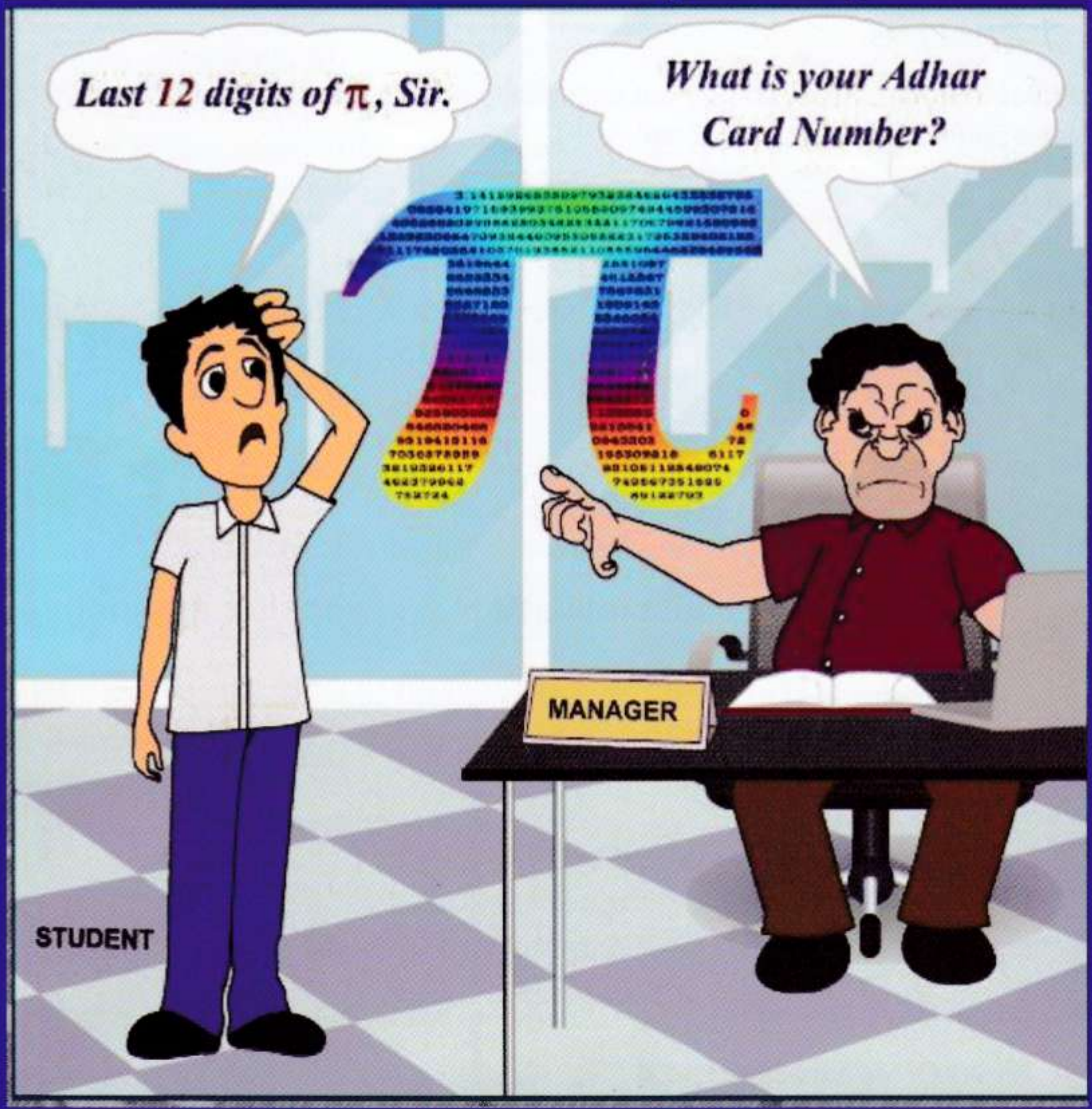
Published by
THE ASSOCIATION OF
MATHEMATICS TEACHERS OF INDIA
CHENNAI, INDIA

Vol. 26 No. 2

DECEMBER 2016

Editor

R. ATHMARAMAN



JM

Published by: The Association of Mathematics Teachers of India (AMTI),
B-19, Vijay Avenue, 85/37, Venkatrangam Pillai Street, Triplicane, Chennai-600005.
Phone: (044)2844 1523; e-mail: amti@vsnl.com and support@amtionline.com

Subscription: Rs. 25/- per year and Rs.250/- for life!

Junior mathematician (JM) is a Mathematics magazine, principally meant for youngsters of age between 8 and 16.

- Aims to interact directly with the fresh, young and receptive minds, motivating them in their appreciation and application of Mathematics.
- Aspires to present Mathematics as a lovable subject, satisfying to the serious-minded and pleasurable for others, removing math-phobia.
- Provokes young student-authors to write their own discoveries and creative thoughts.

Junior Mathematician comes out essentially as a journal for retreat, in a volume of three issues annually as follows:

No.1 in September, No.2 in December and No.3 in March (a double issue to celebrate summer holidays of the school-going)

Articles for the magazine and all correspondence regarding publication material need to be sent to "the Editor: Junior Mathematician" at the address of the AMTI. The Editor is also available on e-mail at athmaramanr@gmail.com and phone 09444611066.

All communications regarding subscriptions, receipt of issues, change of address and other administrative matters should be addressed to "The General Secretary: AMTI, B-19, Vijay Avenue, 85/37, Venkatrangam Pillai Street, Triplicane, Chennai-600005.

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Editorial

SRI. T. DHARMARAJAN

We are deeply saddened to record the sudden demise of a great exponent of "Mathematics for Enjoyment", Sri T. Dharmarajan. A great teacher with a lot of patience, poise and politeness, his simplicity and obliging nature made him a gentleman in every sense. He was closely associated with the AMTI and almost every good Mathematics Learning Project. His fine inputs for the growth of "Junior Mathematician", right from the days of its inception, will always be remembered.

(*Most articles in this issue are contributions of Shri Dharmarajan, sent on various occasions, to Junior Mathematician*).

DIVISOR PUNCH-CARDS

T. Dharmarajan, 2/5/4, Teachers' Colony, Coimbatore -641 022.

Students often are capable of computing, with reasonable speed and ease, the Least Common Multiple (LCM) and the Greatest Common Divisor (GCD, also sometimes called Highest common Factor or HCF) of two or more natural numbers. This is possible for many, even *without fully grasping the 'concepts'* of LCM or GCD.

In what follows, the reader will find the description of a tool that could lead to an understanding of the basic idea of GCD. You might like to call it "Divisor Cards" that resemble computer punch cards.

Take a plain card and punch carefully equally spaced ring-wholes (as shown in (fig.1) to produce a "master template". Each hole represents a prime divisor. The divisors are marked on a margin for easy reference. For example, in the top row of the master template, each hole represents a divisor of 2.

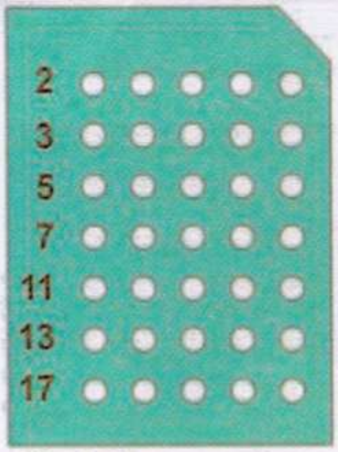


Fig. 1. Master template

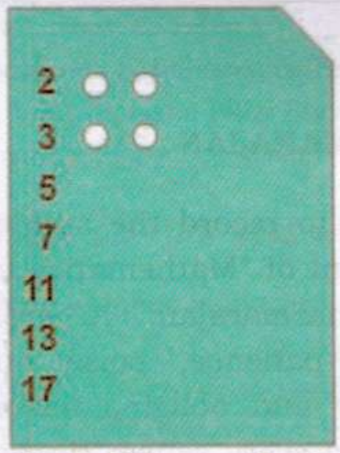


Fig. 2. Divisor card for 36

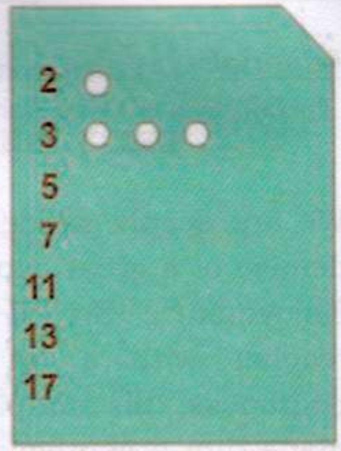


Fig. 3. Divisor card for 54

The master template can now be used to punch cards showing the divisors of various numbers. Suppose you want to visualize the process of finding the GCD of 36 and 54. The card for 36 (fig. 2) has Two 2-hole punches and Two 3-hole punches, representing the prime factorization $2^2 \times 3^2$. Similarly, the card for 54 (fig. 3) would have One 2-hole punch and Three 3-hole punches, since the prime factorization of 54 is 2×3^3.

Now to find the GCD of 36 and 54, place over the card for the first number over the card for the second number. The GCD is then the product of the factors that have common punches. In our case, placing the 36 card on the 54 card and aligning would produce the figure 4. Since One 2 – hole punch and Two 3- hole punches appear , the GCD is 2×3^2 or 18.

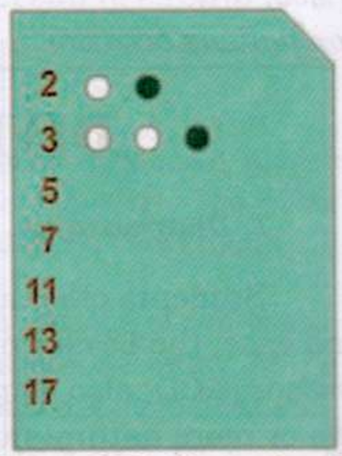


Fig. 4. GCD = 2 x 3 x 3 = 18

You might also like to experiment with three numbers and try to illustrate the GCD in the manner described above.

It is interesting and rewarding to investigate the cases when the card-overlap exhibits the factors for GCD when

(i) one number is a multiple of the other,
(ii) one of them is prime,
(iii) when both of them are primes or
(iv) when they are co-prime.

Start investigating.

DOUBLE, TREBLE OR QUITS

You need two players to play this simple game which is based on the concepts of 'factors and multiples'. Get ready with a "100 Square Board" (that is a 10 × 10 board tessellated with squares labeled from 1 to 100) and also a few counters.

1. Decide on who is to start the game.
2. The two players are to place counters alternately on the empty squares.
3. The first player should begin with placing a counter on a square with an even number.
4. Then the second player has to cover up another square showing a number which is a divisor or multiple of the previous number covered by the opposing player.
5. The players keep on placing counters alternately on empty squares.

The loser is the first player who cannot proceed further in his turn to play. Here is an example:

1st Player	60		36		18		1	
2nd Player		12		9		3		17

The first player now loses. Could the course of action have been cut down cleverly? Is there a winning strategy for either player? Think about these.

The game of Abdul and Amar

A nifty game for you! One player writes a value under Abdul's column while the opponent responds with the same value in a different form using the same digits! Played alternately. Two blanks are there for you to fill up. Solution is available elsewhere in this issue.

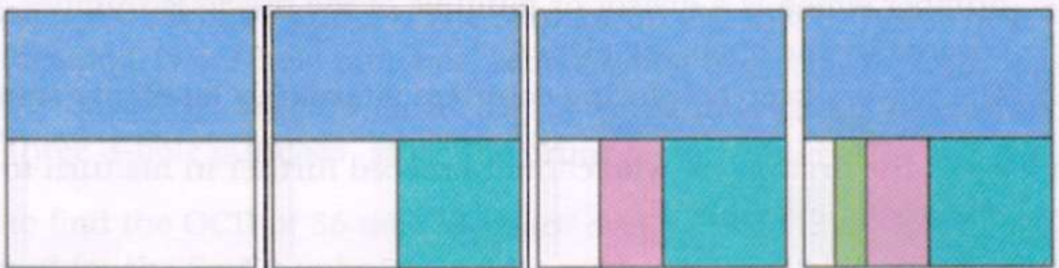
Abdul's Answers	*Amar's counters*
2592	$2^5 9^2$
1+2+3	$1\times2\times3$
$\left(\dfrac{27}{8}\right)^{9/4}$	?
?	343
$\sqrt{121}$	12 - 1
$\sqrt{169}$	$\sqrt{16}+9$ *or* $16-\sqrt{9}$

MANY WAYS OF LOOKING AT A PROBLEM

Sometimes solutions to one single problem can be used to guess and provide solutions of several other similar problems. Visualizing an unknown problem as a case of some known problem is a skill that would be quite useful for us. We look at some examples, where even the known solution (with which we start) is achieved through simple visualization.

- Consider $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots$

Just look at it. It is one-half + one-fourth + one-eighth + ... so on. Being reminded of the way one learns fractions in the elementary school, this can be visualized as summing up of shaded areas of one "whole":



(b) $\frac{1}{2}$ (c) $\frac{1}{2}+\frac{1}{4}$ (d) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ (e) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

Continuing the picture mentally, a reasonable answer for the sum of all the terms we need seems to be 1. (When 'geometric series' is taken up in higher classes, this is easily justified). This is our basic point now.

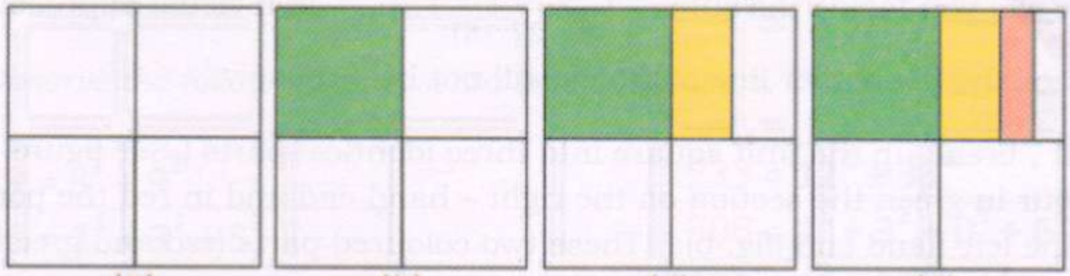
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots = 1 \qquad \ldots\ \ldots\ \ldots\ \text{(i)}$$

- What if you want the sum

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots\ ? \qquad \ldots\ \ldots\ \ldots\ \text{(ii)}$$

The immediate temptation is to compare this with the basic question (i) and notice straightforwardly that the only difference between problem (i) and the new problem is that the term $\frac{1}{2}$ does not appear in the second series. So our answer to the new problem (ii) has to be $= 1 - \frac{1}{2} = \frac{1}{2}$.

One might also visualize this through a chain of figures, as follows:



(a) a unit square divided into four

(b) One-fourth

(c) one-fourth + one-eighth

(d) $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

Extending this visualization further tells us that the required sum is $\frac{1}{2}$.

Alternatively , one could observe that

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots\right]$$

to obtain the required answer.

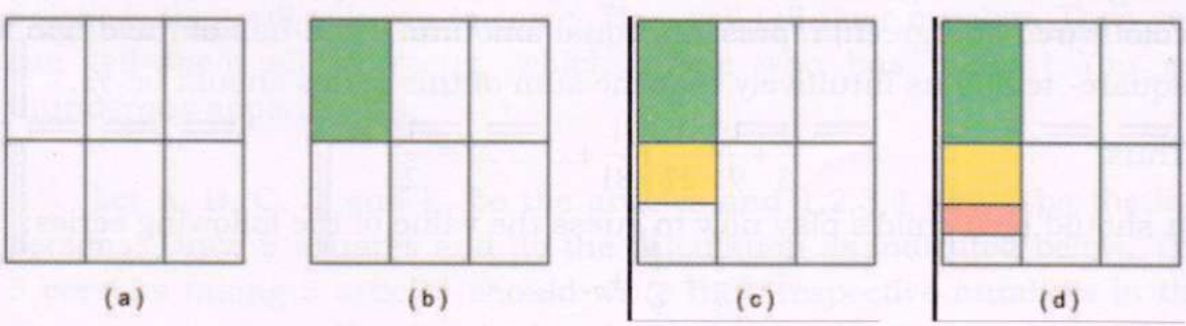
A new question approximating $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots$ is answered with ease now since it is same as question (i) with an extra term 1.

A question like $\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{48}+\ldots\ \ldots\ \ldots = ?$ is after all the result

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{48}+\ldots\ \ldots\ \ldots = \frac{1}{3}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{3}\times 1 = \frac{1}{3}.$$

One can also give a visual justification:



(a) a unit square into 6 equal parts

(b) One-sixth

(c) one-sixth+ one-twelfth

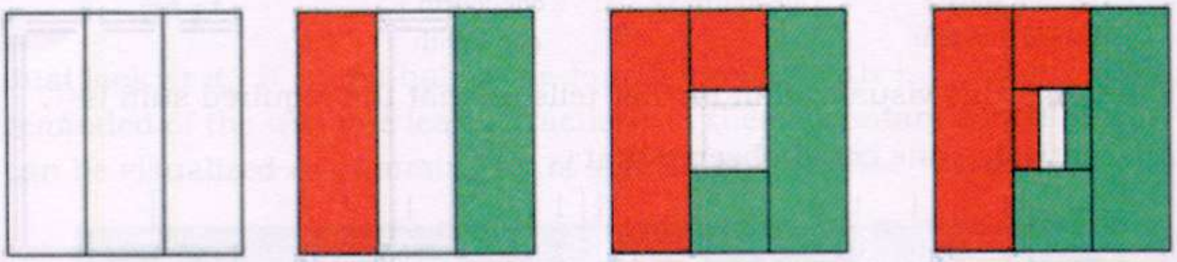
(d) $\frac{1}{6}+\frac{1}{12}+\frac{1}{24}$

and this ultimately leads to the sum $\frac{1}{3}$.

How will you tackle the sum $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots\dots\dots$ in a visual approach?

Obviously, the earlier line of attack will not help us now.

First , break up the unit square into three identical parts (See figure a). Colour in green the section on the right – hand end and in red the portion on the left-hand end (fig. b). These two coloured parts (red and green) have the same area, namely , one-third.



(a) (b) (c) (d)

Consider now the remaining one-third and partition it into three equal parts as shown (each representing one-ninth). This time, colour in green the bottom part and in red the segment on the top (fig . c) . Thus far, we the red and green areas show the same amount, $\frac{1}{3}+\frac{1}{9}$.

Repeat this again, dividing the remaining bit into three identical bits, leaving the middle portion untouched as before and colouring the two end-pieces dividing the remaining bit into three identical bits. We still have equally coloured areas of size $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$.

Imagine that we go on and on till the square is totally filled when each colour (red and green) represents equal amounts – one-half of the original square- telling us intuitively that the sum of this series should be ½.

Thus $$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots\dots\dots=\frac{1}{2}$$

It should be a child's play now to guess the value of the following series:

$$1+\frac{1}{3}+\frac{1}{9}+\dots\dots\dots$$

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\dots\dots\dots$$

$$\frac{1}{6}+\frac{1}{18}+\frac{1}{54}+\dots\dots\dots$$

JMs should try to find such strategies in their encounters with problems.

BEAUTIFUL REPRESENTATIONS

Observe the following representations of numbers. Think of some more!

$89 = 8^1 + 9^2$	$1233 = 12^2 + 33^2$
$135 = 1^1 + 3^2 + 5^3$	$1306 = 1^1 + 3^2 + 0^3 + 6^4$
$198 = 11 + 99 + 88$	$1371 = 1^2 + 37^2 + 1^2$
$283 = 2^5 + 8 + 3^5$	$1386 = 1^4 + 3^4 + 8 + 6^4$
$432 = 4^1 \times 3^3 \times 2^2$	$1676 = 1^1 + 6^2 + 7^3 + 6^4$
$518 = 5^1 + 1^2 + 8^3$	$1715 = 1 \times 7^3 \times 1 \times 5$
$594 = 1^5 + 2^9 + 3^4$	$2213 = 2^3 + 2^3 + 13^3$
$598 = 5^1 + 9^2 + 8^3$	$2354 = 2222 + 33 + 55 + 44$
$746 = 1^7 + 2^4 + 3^6$	$2427 = 2^1 + 4^2 + 2^3 + 7^4$
$952 = 9^3+5^3+2^3 + 9\times5\times2$	$8976 = 8 + 9^4 + 7^4 + 6$

BE A MATHEMAGICIAN

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Keep 5 articles, say, a watch, ring, bangle, hair pin and a ear ring. Keep them on a table. Ask any 5 persons, to take one article each in your absence, and hide it. Then they should do a calculation. All of them will get the same result of one number. Leave the hall. When they have finished, they will ask you to come. They will tell their number. Then you can tell them all, correctly, which article who has taken ! You get thunderous applause !

Let A, B, C, D and E, be the articles and 1,2,3,4 and 5 be the five persons. Draw 5 squares and do the calculation as indicated below. The 5 persons taking 5 articles should write their respective numbers in the respective squares. Number in A to be multiplied by 30, number in B to be multiplied by 1.25, number in C to be multiplied by 0.25. These three to be added. Next take the number in D, multiply it by 2.5 and then the number in E and multiply it by 5. Add these two, separately. Subtract

this total from the previous total. You tell me the result I will tell you who has taken which article. The table below gives the results for various permutations with the serial numbers. Those who can memorize the table can thrill the audience by this magic.

A	B	C	D	E

$$(A \times 30) + (B \times 1.25) + (C \times .25) - \{(D \times 2.5) \;+\; (E \times 5)\}$$

	ABCDE			ABCDE			ABCDE			ABCDE	
1	12345	–1.75	**7**	13245	–.75	**13**	14235	3	**19**	15234	9.25
2	12354	.75	**8**	13254	1.75	**14**	14253	8	**20**	15243	11.75
3	12435	1	**9**	13425	4.75	**15**	14325	5.75	**21**	15324	12
4	12453	6	**10**	13452	12.25	**16**	14325	13.25	**22**	15342	17
5	12534	6.25	**11**	13524	10	**17**	14523	16.25	**23**	15423	17.25
6	12543	8.75	**12**	13542	15	**18**	14532	18.75	**24**	15432	19.75
25	21345	27	**31**	23145	29	**37**	24135	32.75	**43**	25134	39
26	21354	29.5	**32**	23154	31.5	**38**	24153	37.75	**44**	25143	41.5
27	21435	29.75	**33**	23415	37.25	**39**	24315	38.25	**45**	25314	44.5
28	21453	34.75	**34**	23451	47.25	**40**	24351	48.25	**46**	25341	52
29	21534	35	**35**	23514	42.5	**41**	24513	48.75	**47**	25413	49.75
30	21543	37.5	**36**	23541	50	**42**	24531	53.75	**48**	25431	54.75
49	31245	56.75	**55**	32145	57.75	**61**	34125	65.25	**67**	35124	71.5
50	31254	59.25	**56**	32154	60.25	**62**	34152	72.75	**68**	35142	76.5
51	31425	62.25	**57**	32415	66	**63**	34215	68	**69**	35214	74.25
52	31452	69.75	**58**	32451	76	**64**	34251	78	**70**	35241	81.75
53	31524	67.5	**59**	32514	71.25	**65**	34512	83.75	**71**	35412	84.75
54	31542	72.5	**60**	32541	78.75	**66**	34521	86.25	**72**	35421	87.25
73	41235	89.25	**79**	42135	90.25	**85**	43125	94	**91**	45123	106.5
74	41253	94.25	**80**	42153	95.25	**86**	43152	101.5	**92**	45132	109
75	41325	92	**81**	42315	95.75	**87**	43215	96.75	**93**	45213	109.25
76	41352	99.5	**82**	42351	105.75	**88**	43251	106.75	**94**	45231	114.25
77	41523	102.5	**83**	42513	106.25	**89**	43512	112.5	**95**	45312	114.5
78	41532	105	**84**	42531	111.25	**90**	43521	115	**96**	45321	117
97	51234	124.25	**103**	52134	125.25	**109**	53124	129	**115**	54123	135.25
98	51243	126.75	**104**	52143	127.75	**110**	53142	134	**116**	54132	137.75
99	51324	127	**105**	52314	130.75	**111**	53214	131.75	**117**	54213	138
100	51342	132	**106**	52341	138.25	**112**	53241	139.25	**118**	54 231	143
101	51423	132.25	**107**	52413	136	**113**	53412	142.25	**119**	54312	143.25
102	51432	134.75	**108**	52431	141	**114**	53421	144.75	**120**	54321	145.75

HISTORY OF MATHEMATICS IS USEFUL

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Last year during one of the lectures in the Math Club meetings, I learnt a very useful method of solving a particular type of equations.

Suppose you want to find two positive integers whose sum is 28 and the sum of their squares is 400.

Had I been a sixth grade student, I would have attempted like this:

My guess of the two numbers	Their sum	Sum of their squares	OK?
2, 26	28	$2^2+26^2 =$	No
3, 25	28	$3^2+25^2= 634$	No
10, 18	28	$10^2+18^2=424$	No

The last row, third column would tell me that I am somewhere near a good guess and I would go on guessing till I get the correct fit.

In my ninth grade, I would have tried to form equations. Let x , y be the required numbers. Then I need to solve

$$x + y = 28 \quad \dots \quad \dots \quad \dots \quad (1)$$
$$x^2 + y^2 = 400 \quad \dots \quad \dots \quad \dots \quad (2)$$

(2) is same as $x^2 + (28 - x)^2 = 400$ which simplifies to

$$2x^2 - 56x + 384 = 0,$$

or $$x^2 - 28x + 192 = 0.$$

Using the formula for solving quadratic equations,

$$x = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(1)(192)}}{2\times 1}$$
$$= \frac{28 \pm \sqrt{16}}{2} = \frac{28 \pm 4}{2} = 16 \text{ } or \text{ } 12$$

We can easily verify if our answers 16 and 12 are correct.

Consider now the method suggested by *Diaphantous* (Alexandrian Greek mathematician of probably 3rd century CE) given here:
Let the required positive integers be $14 + z$ and $14 - z$.

Then by the given conditions,

$$(14 + z)^2 + (14 - z)^2 = 400.$$

Simplifying, this is $2\,(196 + z^2) = 400$ which is same as $z^2 = 4$ or $z = \pm 2$. You get the answers 16 and 12 from $14 + z$ and $14 - z$.

Will this course of action work in the general case?

Suppose you want to find two positive integers whose sum is p and the sum of their squares is q. (Should $q > p$? Think about it.)

The routine method will provide the equations

$$x + y = p \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$x^2 + y^2 = q \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Replacing the value $y = p - x$ in (4) and simplifying, we obtain

$$2x^2 - 2px + (p^2 - q) = 0$$

Using quadratic formula, the solutions are given by

$$x = \frac{-(-2p) \pm \sqrt{(-2p)^2 - 4(2)(p^2 - q)}}{2\times 2}$$

$$= \frac{2p \pm \sqrt{8q - 4p^2}}{2\times 2} = \frac{p \pm \sqrt{2q - p^2}}{2}$$

Diaphantous' trick would be to assume the solutions to be of the form $\frac{p}{2} + z \text{ and } \frac{p}{2} - z$, so that $\left(\frac{p}{2} + z\right)^2 + \left(\frac{p}{2} - z\right)^2 = q$. Simplifying this, $2z^2 = q - \frac{p^2}{2}$

or $$z = \pm\sqrt{\frac{q}{2} - \frac{p^2}{4}} = \pm\frac{\sqrt{2q - p^2}}{2}.$$

The final answers as before are from $\frac{p}{2} + z \text{ and } \frac{p}{2} - z$. This simpler procedure of *Diaphantous* is part of History of Mathematics which many of us ignore to study. In our country *Brahmagupta, Bhaskara, Madhava* and many mathematicians of ancient days have provided mathematically rich resources worth exploring. There is a lot to learn from the ancients.

Do you think that Diapantous method will work even if the numbers to be found are not integers? Or is there any other constraint? JMs should start investigating.

CONJECTURING

Anirudha C.P., 3R-17, Sainik School, Bhubaneshwar-751005 (Odisha)

I started looking at Factorials, simply because they are a fascinating way of representing numerical values. However, while exploring some of them quite recently, I found some quite interesting characteristics, which I share here with the readers.

The factorial $n!$ is defined for a positive integer n as

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$$

In practice, this implies multiplying a series of descending positive integers. For example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$1! = 1$$

There are standard algebra books which one can refer to study in detail about this notation.

Observe 6! $\quad 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$$= (3 \times 2) \times 5 \times 2^2 \times 3 \times 2$$

$$= 5 \times 3^2 \times 2^4$$

which could be re-written as

$$\boxed{6! = 5^{2^0} \times 3^{2^1} \times 2^{2^2}} \quad \ldots\ldots\ldots (1)$$

Similarly take a look at 10!

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= (5 \times 2) \times 3^2 \times 2^3 \times 7 \times (3 \times 2) \times 5 \times 2^2 \times 3 \times 2$$

$$= 7 \times 5^2 \times 3^4 \times 2^8$$

And this is same as

$$\boxed{10! = 7^{2^0} \times 5^{2^1} \times 3^{2^2} \times 2^{2^3}} \quad \ldots\ldots\ldots (2)$$

Study the formation of the product on the right sides of (1) and (2) and they are really beautiful.

I searched for such positive integers up to 20!, working out patiently each every factorial. But I did not find any other natural number with the characteristic property exhibited by (1) and (2).

So I conjectured that the expression

$$n! = p_o^{2^0} \times p_1^{2^1} \times p_2^{2^2} \times \ldots\ldots\ldots 5^{2^{m-3}} \times 3^{2^{m-2}} \times 2^{2^{m-1}}$$

where *i)* n and m are natural numbers

ii) $p_o, p_1, p_2, \ldots$ are conseqcutively decreasing primes less than n and

iii) $m = \Pi(n) = \Pi(p)$, Π being prime counting function,

has only two solutions, namely $n = 6$ and $n = 10$.

On a later date, when I revisited the conjecture, I stumbled upon an idea of obtaining squares on the right side of (1) and (2). From (1),

$$6! = 5^{2^0} \times 3^{2^1} \times 2^{2^2}$$

$$6! \times 5 = 5 \times 5^{2^0} \times 3^{2^1} \times 2^{2^2}$$

$$= (5 \times 4 \times 3)^2$$

$$\boxed{\therefore \quad 6! \times 5 = 30^2}$$

In the same way,

$$10! = 7^{2^0} \times 5^{2^1} \times 3^{2^2} \times 2^{2^3}$$

$$10! \times 7 = 7 \times 7^{2^0} \times 5^{2^1} \times 3^{2^2} \times 2^{2^3}$$

$$= (7 \times 5 \times 9 \times 16)^2$$

$$\boxed{\therefore 10! \times 7 = 5040^2}$$

Now I am tempted to refine my conjecture that

$$n! \times p = m^2$$

(where m, n are natural numbers and p is the largest prime $\leq n$) has only two solutions, namely $\{n = 6,\ m = 30\}$ and $\{n = 10 \text{ and } m = 5040\}$.

Am I going right on conjecturing? Can my conjectures be proved? I am interested to get a feedback from my esteemed JM colleagues.

FALLACIES

A fallacy is a misleading argument or belief based on faulty knowledge leading to error in reasoning. Here we see some mathematical fallacies.

1 . Take a thin cardboard of side length 8 cm. Subdivide it as shown in the figure (i) and re-arrange it as a rectangle as shown in figure (ii).

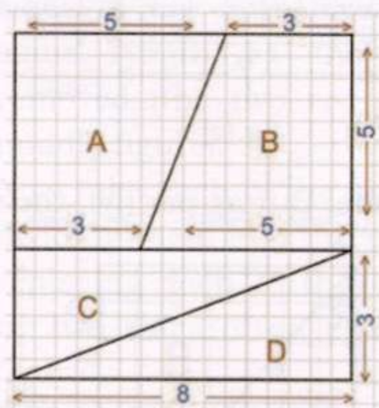


fig.(i)

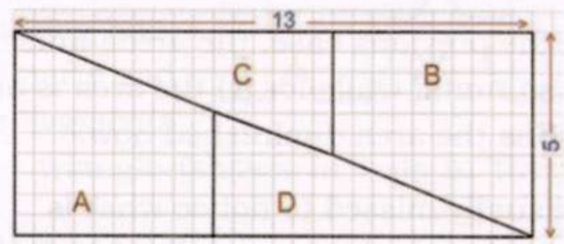


fig.(ii)

Both the e figures should be equal in area because (ii) is only a different rearrangement of (i). But you find by calculation that the area of fig (i) is $8 \times 8 = 64$ square units, but that of fig (ii) is $13 \times 5 = 65$ sq.units. Where does the extra 1 sq.unit come from?

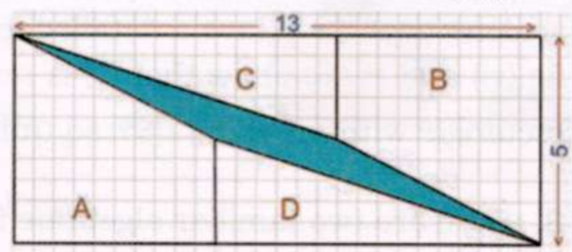


fig.(iii)

The fallacy arises due to inaccurate visualization. In reality, the pieces do not fit together as shown in figure (ii). There is a long, narrow parallellogram-like gap, the area of which is 1 sq.unit.

[See fig.(iii) where the shape is shown duly enlarged. Try to obtain the area of the gap, using Pythagoras theorem].

2. What follows is a similar fallacy, but slightly different. Draw on a square card-board a 64 square (chess-board like) pattern. Make two straight cuts along it, one long and one short, thereby partitioning the board into three different pieces, namely A, B and x. See fig.(iv). Slide the piece B upwards along A; remove the tiny triangular piece x to fit into the upper left corner. The pieces fit together nicely as in fig.(v).

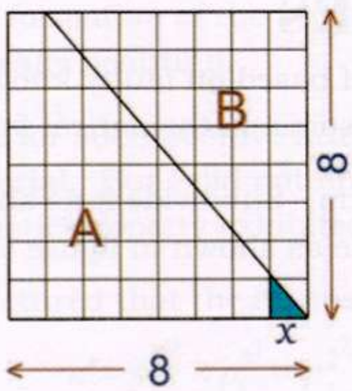


fig.(iv)

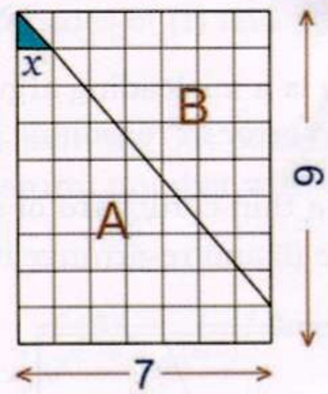


fig.(v)

As in the previous case, the areas must be same. But as per (iv), the area is 64 sq.units, while for (v) the area is only 63. Has the area diminished now?

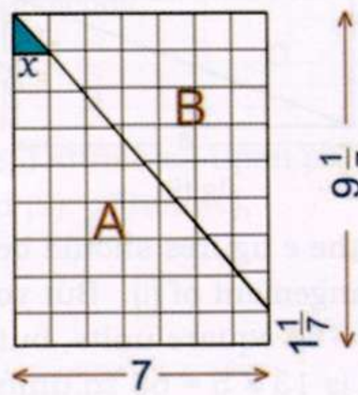


fig.(vi)

You visually missed to notice what is happened to the horizontal lines in fig.(v). Did you see that the sliping does not cut across the bottom right square diagonally but breaks into the next square to form a scalene triangle. Have a detailed look at fig.(vi) and try to show that (v) is really a rectangle with an area $9\frac{1}{7}\times 7$ sq.units.

3. Here are two familiar algebraic fallacies, which you can easily solve:

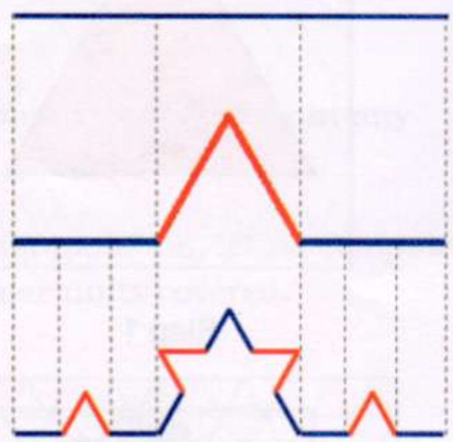
$$
\begin{aligned}
\textit{Let}\quad a&=b\\
a^2&=ab\\
a^2-b^2&=ab-b^2\\
(a+b)(a-b)&=b(a-b)\\
a+b&=b\\
2b&=b\\
2&=1
\end{aligned}
$$

Note Step 5!

$$
\begin{aligned}
\textit{Let}\quad x&=\frac{\pi+3}{2}\\
2x&=\pi+3\\
2x(\pi-3)&=(\pi+3)(\pi-3)\\
2\pi x-6x&=\pi^2-9\\
9-6x&=\pi^2-2\pi x\\
9-6x+x^2&=\pi^2-2\pi x+x^2\\
(3-x)^2&=(\pi-x)^2\\
3-x&=\pi-x\\
3&=\pi
\end{aligned}
$$

Note Step 8!

THE SNOWFLAKE CURVE



- Begin with a st. line (shown in blue at the top).
- Divide it into three equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length as the segment being removed (the two red segments in the middle figure).
- Now take each of the four resulting bits, dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle (shown in reds in the bottom figure).
- Repeat this process again and again.

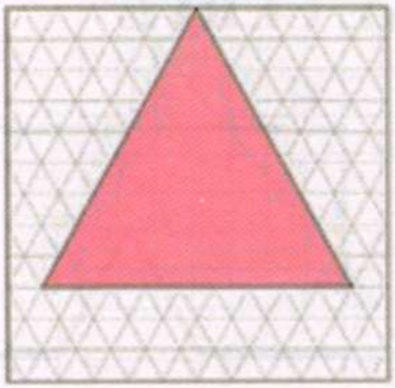
  This process (known as iterative process) gives you a curve. This leads to the construction of what is known as a Snowflake curve.

The Snowflake curve is obtained as follows:

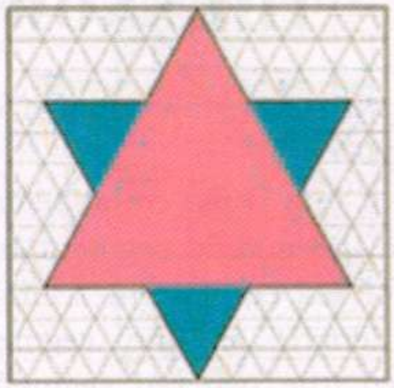
1. Draw an equilateral triangle.
2. Divide each side into three equal parts.
3. On the middle part of each side, draw an equilateral triangle (to the *outside* of the original triangle).
4. Wipe away the line joining the lines connecting the new triangles to the large one. (Note that the original triangle has sides of three parts and the next figure grows having sides of four parts).
5. Repeat step 2 for *all* the lines you now have in your figure.

   You have started to get a Snow-flake triangle; and you can repeat the steps (called *iterations*) any number of times.

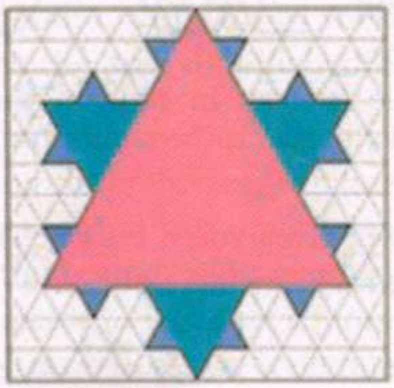
What does it look like? See a few steps below:



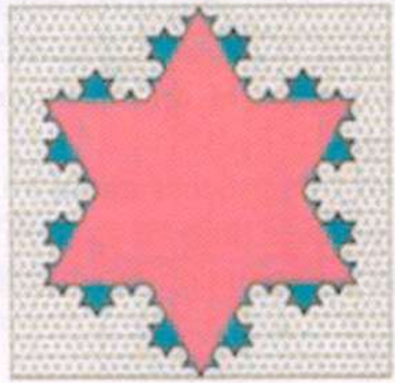
Step 1



Step 2



Step 3



Step 4

The Snowflake curve is not only beautiful in appearance, but also provides fascinating results when we consider its perimeter and area.

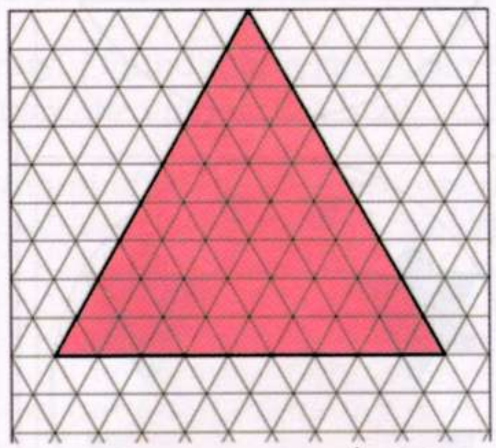
What is the perimeter of the curve?
At every step, we replace a line segment of length 1 unit with a line segment of $\frac{4}{3}$ units. (See figure).

In our figure, the starting position has length 9 units. So, how will the perimeter go on changing?

Steps	1	2	3	4	5	6	7	8	.	.	.	.	.	.
Lengths	9	$9\times\frac{4}{3}$ $=12$	$12\times\frac{4}{3}$ $=16$	$16\times\frac{4}{3}$ $=21\frac{1}{3}$	$\frac{64}{3}\times\frac{4}{3}$ $=28\frac{4}{9}$	.	.	.	.	.	.	.	.	

Can you find the subsequent lengths? Do you see a pattern?
Can you predict the number of iterations it would it take to reach a perimeter close to 100 units?
Can you generalize?
Can you find the length at the n^{th} step?
The list giving the lengths, step by step, goes on increasing without any limit*!*

How does the *area* aspect behave? For easy manipulation, let us compute the area counting the number of small triangular units covered.



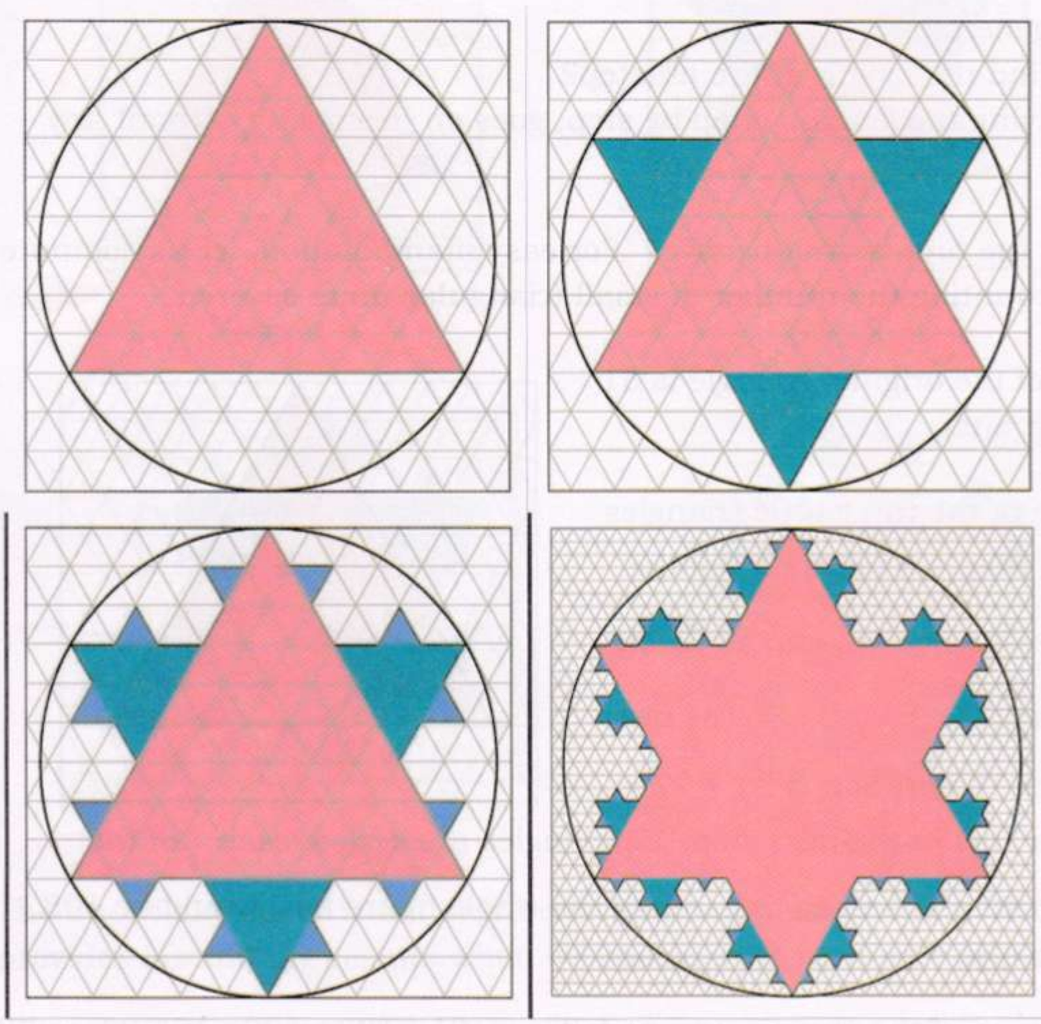
The area of the original triangle is 81 (triangular bits).

Now, each of the three little triangles we add in Step 1 has a side $\frac{1}{3}$ of the original and hence an area of $\frac{1}{9}$ of it, namely each $\frac{1}{9} \times 81 = 9$. So the new figure after 1st iteration is 81 + 27 = 108. After this in the next step, 12 triangles are added (each $\frac{1}{9}$ of the triangle in step 1) and so the area of the whole figure now would be 108 + $12 \times \frac{1}{9} \times 9 = 120$.

Thus the area of the curve grows as follows: 81, 108, 120, 125.33, 127.7, 128.75, 129.21, 129.43, 129.522, ...

One can notice that the growth slows down as the steps increase. A patient reader can work out further and find that the values go closer and closer to some constant.
If you draw the circum circle of the original triangle, then irrespective of the number of steps you go (i.e., irrespective of the number of iterations), the whole curve and its region lies within that circle. (See figures that follow).
No matter how large the perimeter gets, the area of the figure remains inside the circle.

That is, in the case of Snowflake, an infinite perimeter encloses a finite area. What a wonder!



There are many more surprises about this curve. We have just begun to understand it. JMs will do well to extend their learning further by going through good books in the library that deal with Snowflake.

A Puzzle: You have 12 matches of unit length. Use all of them, and try to make a polygon whose area is 3 square units.
(*Solution is given elsewhere in this issue*)

GRAPHING VARIETIES

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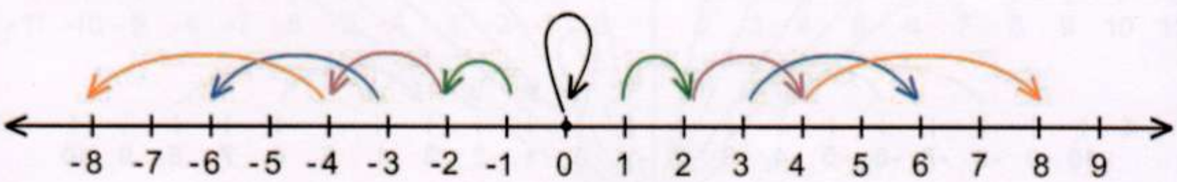
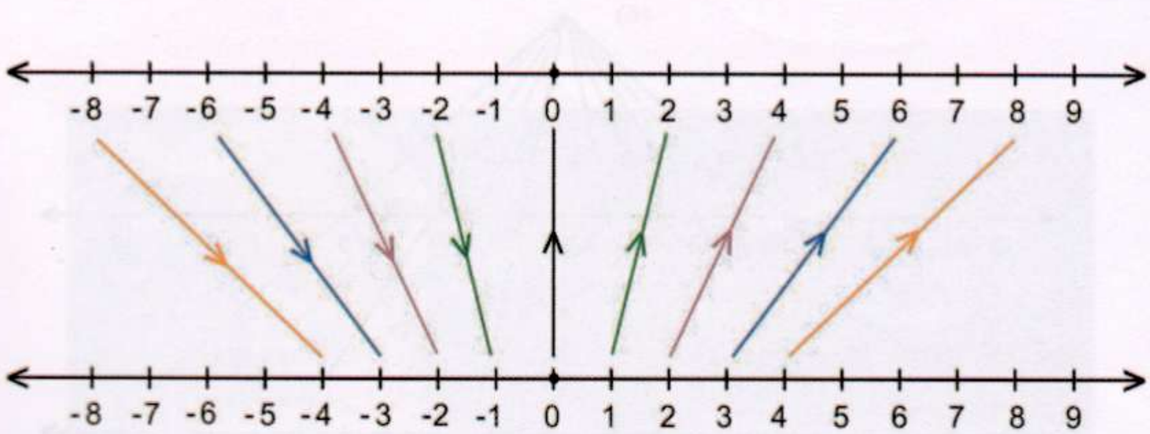
Graphs can serve many purposes: to visually read data, to compare and contrast details, to show relations (mostly given in algebraic terms), etc. But, whenever I thought of a graph in mathematics, I associated it with two perpendicular lines called x axis and y axis and one or two lines drawn on the paper, intersecting or parallel, giving some interpretation in terms of mathematical ideas. Many times I have been questioning myself: why should the x and y axes be perpendicular to one another? Is this the only way to show the mathematical relation?

During one of my recent visits to the library, I found that a graph can be in different forms and styles; I want to share the approaches here.

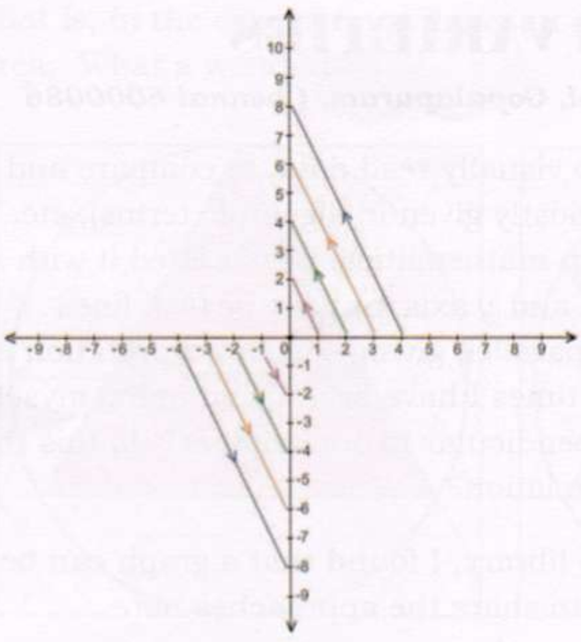
For simplicity, let us consider the easiest form of relation $y = mx$, where we are concerned with real numbers; as an example we take $y = 2x$. This can be shown in many different ways:



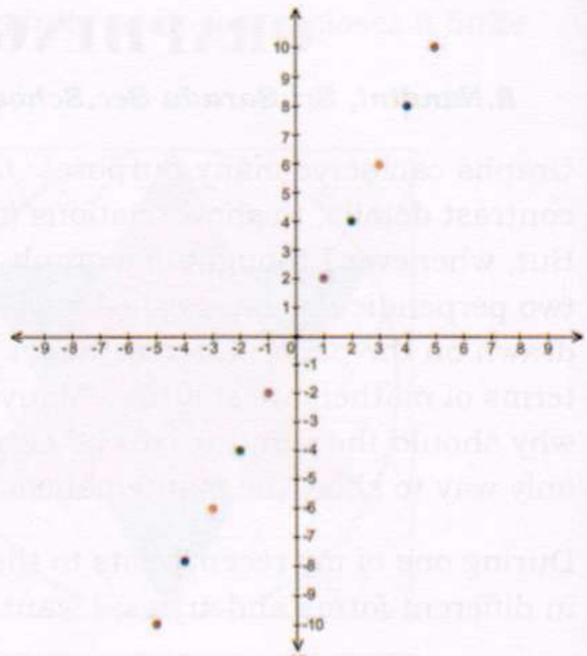
(a)



(b)

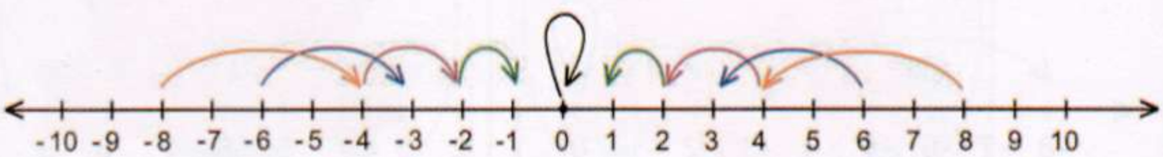


(c)

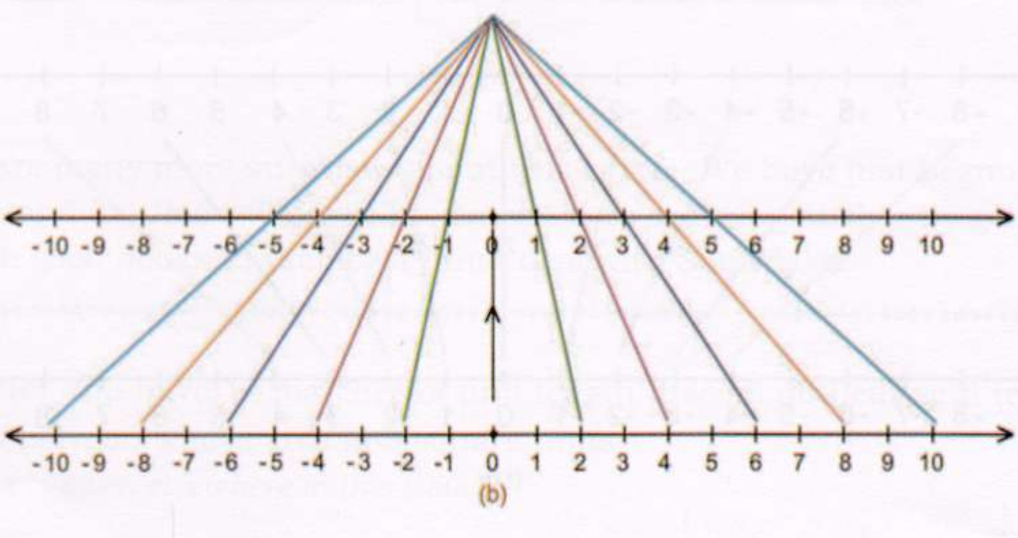


(d)

When I tried to draw similar graphs for $y=\frac{1}{2}x$, I found in picture (b) that the connecting lines always meet at a point. I also understood the significance of inserting directed arrow marks in this case.

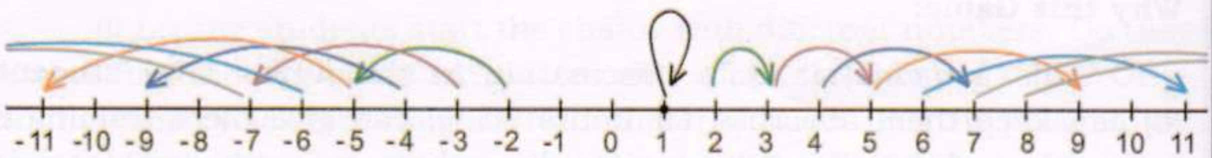


(a)

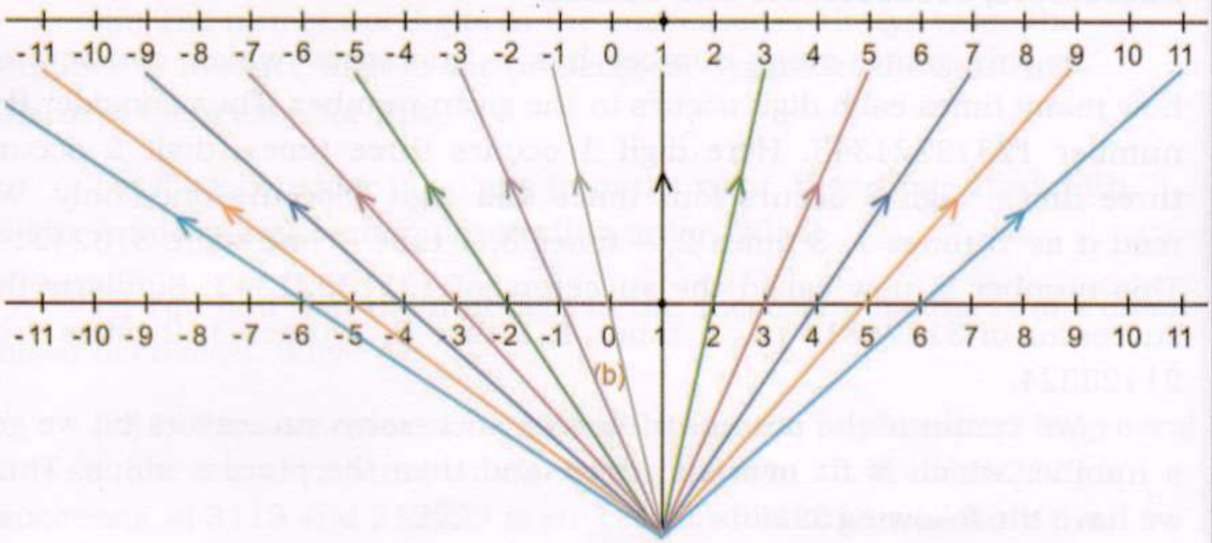


(b)

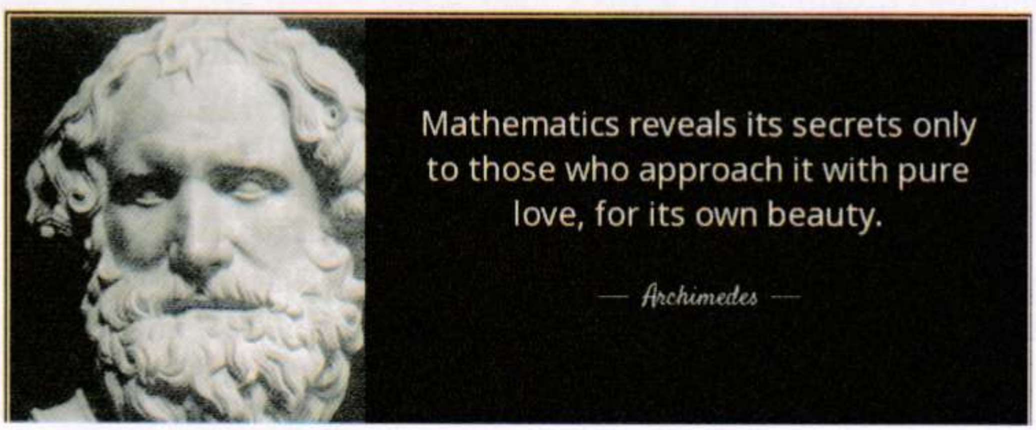
I guess that this property of concurrence at a point should always be the case in such graphs. To assure myself, I drew the graph of $y = 2x - 1$. The graphs I have experimented with are given here for comments from readers.



(a)



(*b*)



A NUMBER GAME FOR JUNIORS

(A favourite piece of writing by Dr. J.N. Kaour, one of AMTI's former Presidents; he wrote this in 1978 but still could be a motivating article)

Why this Game:

This game (a) is quite fascinating to the junior level students (b) can keep them absorbed for hours on (c) can give them training in recognition of patterns (d) can introduce them to some combinatorial mathematics (e) can introduce them to elementary logical reasoning and (f) can give rise to many 'open' problems.

Successors, Predecessors and Chains:

In this game, every number has a 'successor' which enumerates how many times each digit occurs in the given number. Thus consider the number 12313221343. Here digit 1 occurs three times, digit 2 occurs three times, digit 3 occurs four times and digit 4 occurs once only. We read it as '3 times 1, 3 times 2, 4 times 3, 1 time 4' and write 31324314. This number is now called the successor of 12313221343. Similarly the successor of 31324314 is 2 times 1, 1 time 2, 3 times 3, 2 times 4 or 21123324.

We continue the process of finding successive successors till we get a number which is its own successor and then the process stops. Thus we have the following 'chains'.

1	4	7	9	13
11	14	17	19	1113
21	1114	1117	1119	3113
1112	3114	3117	3119	2123
3112	211314	211317	211319	112213
211213	31121314	31121317	31121319	312213
312213	41122314	41122317	41122319	212223
212223	31221324	3122131317	3122131419	114213
114213	21322314	4122231417	4122231419	31121314
31121314	21322314	3132132417	3132132419	41122314
41122314		3122331417	3122331419	31221324
31221324		3122331417	3122331419	21322314
21322314				21322314
21322314				

The number of which a given number is the successor is called its 'predecessor'.

Patterns Observed

(i) Let the students start the chains with different numbers. Do they always find that the chain terminates after a finite number of steps? Or do they find cycles repeating after a number of steps. (For the present, we consider only those starting numbers in which the digit 0 does not occur).

(ii) Except possibly in the first number, the number of digits in every successor number is even. Why?

(iii) The number of digits in the successor is always twice the number of 'distinct' digit in the predecessor. Again let the students discover the reason for this.

(iv) If we consider the digits from the right, then first, third, fifth...... digits from the right are in descending order, Why?

(v) You find that the unit digit in the successive members of a chain never decreases. Why?

(vi) Is the successor always greater than the predecessor? In general it is so, but it is always not so. Eg. In the chain of 13. We find 2123 is successor of 3113 and 212223 is successor of 312213. Similarly the successor of 888888888 is just 98.

(vii) Each number has a unique successor. Eg. The predecessor of 21322314 must have digit 1 twice, digit 2 three times, digit 3 two times and digit 4 once, but these may be in any order so that some of the possible predecessors are 11222334, 12341232, 43232211.

(viii) Many chains end with the same number. Eg. in the chain of 1, 4 or 13 above, the last three numbers are the same.

(ix) Each chain has a number of steps. In each step, the number of digits is the same. Within each steps, the sum of the odd-numbered digits counted from the left is the same.

Two numbers are said to be 'similar' or 'isomorphic' if they have the same digits repeated the same number of times: the order of the digits need not be the same. If the order is also the same, the numbers become identical.thus 12 and 21 are similar and 1112, 2111, 1211, 1121 are also similar. The readers can easily verify that

(a) every number is similar to itself.

(b) if a given number is similar to another number than the second number is similar to the given number.

(c) if, out of three given numbers, the first number is similar to the second and the second number is similar to the third, the first number is similar to the third.

Thus similarity of numbers is an equivalence relation and all numbers can be divided into equivalence classes such that numbers belonging to a class are similar to one another and no number of one class is similar to any number of another class.

How many numbers are similar to a given number?

Consider 123:

Numbers similar to its are 123,132, 231, 213, 312, 321 and these are $6 = 1 \times 2 \times 3 = 3!$ in number. (See about $n!$ elsewhere in this issue).

Consider 1234:

Numbers similar to it are 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2341, 2414, 2413, 2431, 3142, 3124, 3214, 3241, 3412, 3421, 4132, 4123, 4231, 4213, 4312, 4321.

Their number is $24 = 1 \times 2 \times 3 \times 4 = 4!$ In general if a number has n distinct digits, there are $n!$ numbers similar to it, including itself. Similarly numbers similar to 113 are 113,131,311 and these are $\left(\frac{3!}{2!}=\right)$ 3 in number. By doing a number of such examples and generalizing, the readers can see that the number of numbers similar to a given number of n digits in $n!/(n_1!\, n_2! \ldots n_r!)$ where n_1 digits are same, n_2 digits are same and so on.

The importance of similar numbers in our game in this that similar numbers give rise to the same successor and the number of predecessors of a given number is obtained by forming the number of numbers similar

1

11

21 12

1112 1121 1211 2111

3112 3121 3211 1123 1132 1231 1312 1321 2113 2131 2311

to any of its predecessors.

The consideration of similar numbers enables us to deduce number of chains from a given chain. Thus the first chain gives us the chains for the following numbers.

The total number of chains deducible from first chain is

$$1+\frac{2!}{2!}+\frac{2!}{1!1!}+\frac{4!}{3!}+\frac{4!}{2!}+\frac{6!}{3!2!}+\frac{6!}{2!2!2!}+\frac{6!}{4!}+\frac{6!}{3!}+\frac{8!}{4!2!}+\frac{8!}{3!2!2!}+\frac{8!}{2!3!2!}$$

$$= 1+1+2+4+12+60+90+30+120+840+1680+1680+1680$$

$$= 6200$$

The readers may like to find chains for all numbers up to say 9999. They write all numbers for 1 to 9999 (omitting numbers including zeros) on a piece of paper. There are $9 \times 9 \times 9 \times 9 = 6561$ such numbers (Why?) they find the chain for 1 and cancel 1, 11, 21,12, 1112, 1211, 1121, 2111, 3112, 3121, 3211, 1123, 1132, 1231, 1213, 1312, 1321, 2113, 2131, 2311 from the list. Similarly they find the chains for 2, 3, 4 ....

They stop any chain as soon as they reach a number which has already been crossed.

Numbers which have no Predecessors.

Every number need not have a predecessor. Some examples are the following:

(i) All odd digits numbers (Why?)

(ii) No numbers in which four successive digits are the same (Why?)

(iii) Numbers in which odd numbered digits from the right are not decreasing.

(iv) Numbers in which the sum of the odd-numbered digits from the left is odd can have immediate predecessor but cannot have second generation predecessor.

Numbers which are their Successors

The smallest such number is 22; other examples we have come across 21322312, 31223314, 31223315, 31223316, 31223317, 31223318, 31223319. Of course every such number has to have an even number of digits (why ?). The only number of two digits which is its own successor is 22.

Can there be a number of four digits which is its own successor? Since the number of digits in it have to be the number of distinct digits in the predecessor (i.e. it must be of the form *aaab* or *aabb*. The successor of *aaab* is $3a1b$ and since it can have also distinct digits. There are only four possibilities viz. 3311 or 3311 or 3111 or 3313.

All these fail to satisfy our conditions. Similarly the successor of *aabb* is $2a2b$ and all the possibilities in this case can also be ruled-out. Thus there is no four digit number which is its own successor.

Some open Problems:

The students may attempt the following problems:

(a) Either prove that all numbers have finite chains or find one number which does not have such a chain;

(b) Find all 6, 8, 10, 12, ... 20 digit numbers which are their own successors.

(c) How many distinct chains are there for numbers less than
(i) 9999 (ii) 999999

(d) Investigate the problems if 0 is allowed to be a digit.

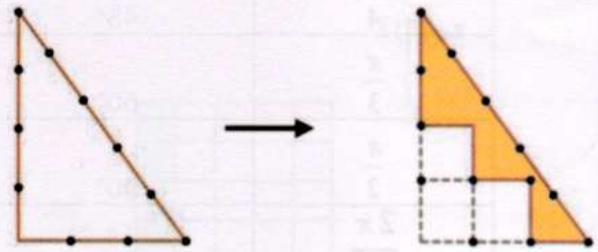
Answers to Puzzle:

1. Abdul or Amar? Answers can vary since there are many possibilities. A sample is here:

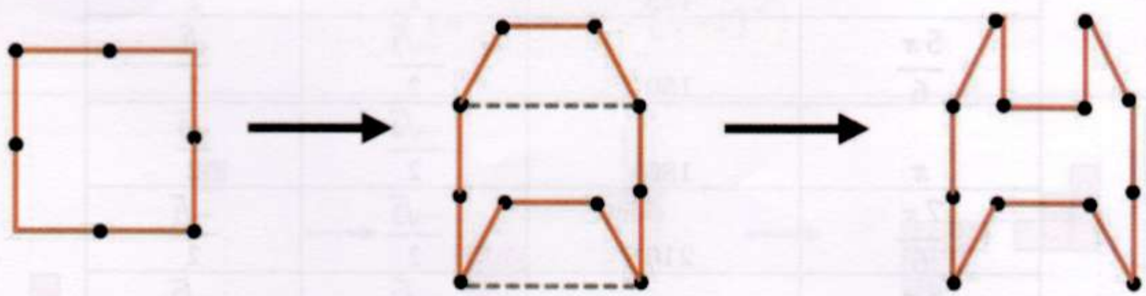
Abdul's Answers	*Amar's counters*
$\left(\frac{27}{8}\right)^{9/4}$	$\left(\frac{9}{4}\right)^{27/8}$
$(3+4)^3$	343

2. Matchstick puzzle:

There are several ways of doing it. Make a 3-4-5 triangle using all the 12 matchsticks. (Its area = 6 sq.units). Now move 4 matches into the triangle as shown in the diagram. In the process, you find the new (polygonal) area is 3 sq.units!



A second method is as follows:

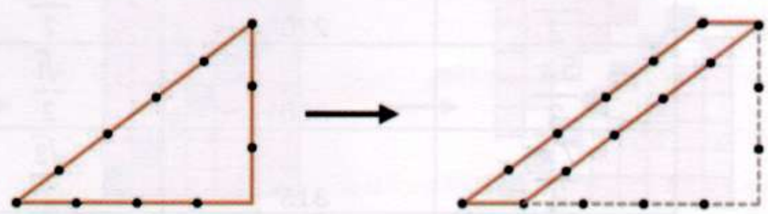


Begin with a 4 sq.unit square.

In this conversion, there is no change in the area.

Last transformation leaves you with area 3 sq.units.

This last one is straightforward and quite simple.



What is **HEXAKOSIOIHEXEKONTAHEXAPHOBIA** ?

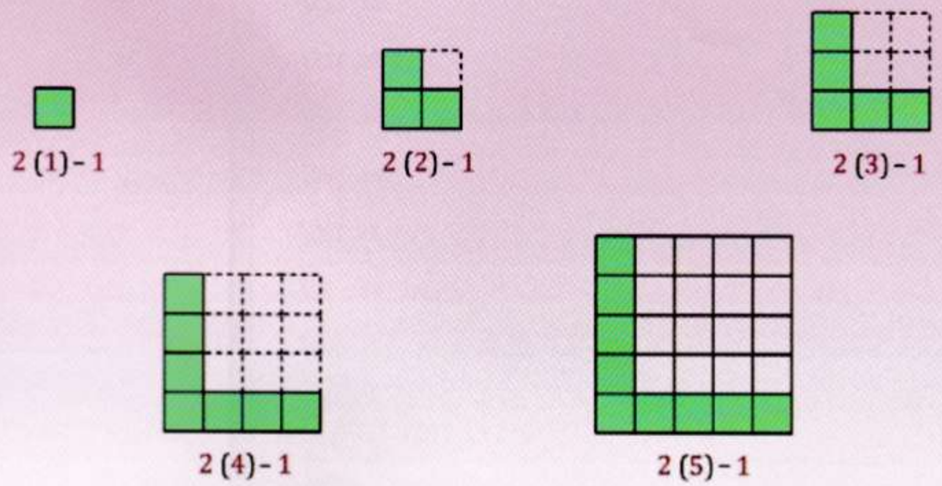
It means:

Fear of the Number 666.

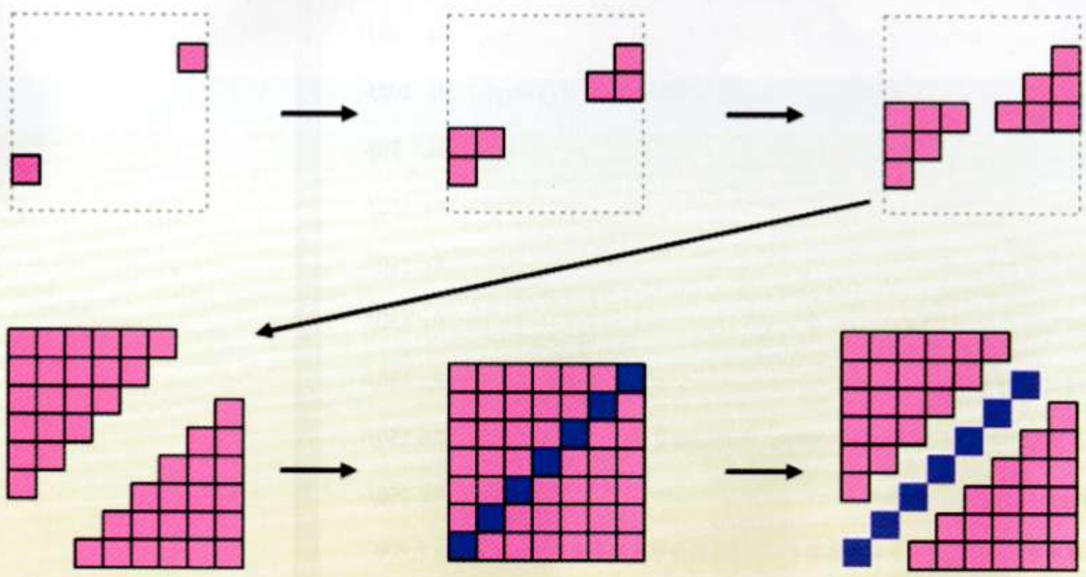
A USEFUL TABLE FOR TRGIGONOMETRY

x *(radians)*	*x* *(degrees)*	*cos x*	*sin x*
0	$0°$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$
$\frac{\pi}{6}$	$30°$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$90°$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$
$\frac{2\pi}{3}$	$120°$	$\frac{-\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$135°$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$150°$	$\frac{-\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$
π	$180°$	$\frac{-\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$
$\frac{7\pi}{6}$	$210°$	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{1}}{2}$
$\frac{5\pi}{4}$	$225°$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$240°$	$\frac{-\sqrt{1}}{2}$	$\frac{-\sqrt{3}}{2}$
$\frac{3\pi}{2}$	$270°$	$\frac{\sqrt{0}}{2}$	$\frac{-\sqrt{4}}{2}$
$\frac{5\pi}{3}$	$300°$	$\frac{\sqrt{1}}{2}$	$\frac{-\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$315°$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$330°$	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{1}}{2}$
2π	$360°$	$\frac{\sqrt{4}}{2}$	$\frac{-\sqrt{0}}{2}$

Maths Through Pictures



$$2\ (n-1) = n^2 - (n-1)^2$$



$$1+2+3+\ldots.+n = \frac{1}{2}\left[(n+1)^2 - (n+1)\right] = \frac{1}{2}n^2 + \frac{1}{2}n$$

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